Mathematics for Computer Science

TD3

September 24^{th}, 2025

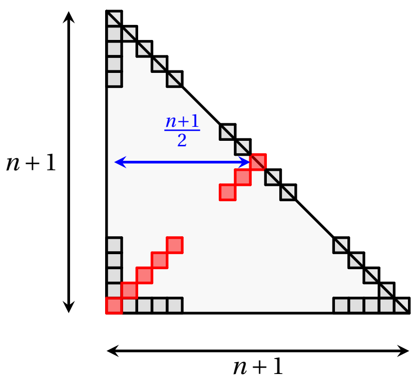
Question. *Prove that Fibonacci numbers are sums of anti-diagonals in Pascal's Triangle.*

Proof. Let us define, for all $n \in \mathbb{N}$, $D_n \overset{\text{def}}{=}$ "*sum of the n^{th} anti-diagonal*" (starting at $n = 0$ for the first anti-diagonal). We have, for the first terms:

n	Sum	Number of terms	D_n
0	1	1	1
1	1	1	1
2	$1+1$	2	2
3	$1+2$	2	3
4	$1+3+1$	3	5
5	$1+4+3$	3	8
6	$1+5+6+1$	4	13
$\vdots$	$\vdots$	$\vdots$	$\vdots$

1								
1	1							
1	2	1						
1	3	3	1					
1	4	6	4	1				
1	5	10	10	5	1			
1	6	15	20	15	6	1		
1	7	21	35	35	21	7	1	
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$

We observe that the number of terms follows a pattern, from which we can conjecture that the sum D_n is consistituted of $\lfloor n/2 \rfloor + 1$ terms. This observation comes from the fact that, if we fix $n \in \mathbb{N}$, we can build a *reduced Pascal Triangle* of size $(n+1)$ by $(n+1)$ consitituted of the first $(n+1)$ rows of the Pascal's Triangle.



On this figure, we represented in red the coefficient we need to add to build D_n. Geometrically, the number of terms is the horizontal distance between the origin of the drawing (*i.e.* the bottom-left square) and the middle on the diagonal of the drawing. This distance, figured in blue, is exactly $\dfrac{n+1}{2}$.

The number of terms is an integer k, and in the case where the last red square overlaps the diagonal (which is what happens on the figure), this number is exactly such that $\dfrac{n+1}{2} \leq k$, or equivalently, $k = \left\lceil \dfrac{n+1}{2} \right\rceil$. Equivalently, we will write this quantity as

$$\left\lceil \frac{n+1}{2} \right\rceil = \left\lfloor \frac{n+1}{2} + \frac{1}{2} \right\rfloor = \left\lfloor \frac{n}{2} + 1 \right\rfloor = \left\lfloor \frac{n}{2} \right\rfloor + 1.$$

Now, making our sum start at index 0, our sum expresses as

$$D_n = \sum_{k=0}^{\lfloor n/2 \rfloor} \binom{n-k}{k} = \binom{n}{0} + \binom{n-1}{1} + \cdots + \binom{n - \lfloor n/2 \rfloor}{\lfloor n/2 \rfloor}.$$

We will now prove by induction that for all $n \in \mathbb{N}$, $D_n = F(n)$ the n^{th} Fibonnaci Number.

- ▷ *Base case* : as summarised in the first tabular, we indeed have $D_0 = 1 = F(0)$ and $D_1 = 1 = F(1)$.
- ▷ *Induction step* : let $n \in \mathbb{N}$ be such that $D_n = F(n)$ and $D_{n+1} = F(n+1)$. Now, depending on whether n is even or odd, we will show that $D_{n+2} = F(n+2)$.

- If n is even, let us write $n = 2m$ for some $m \in \mathbb{N}$. We observe that $\lfloor (n+2)/2 \rfloor = m+1$, $\lfloor (n+1)/2 \rfloor = m$ and $\lfloor n/2 \rfloor = m$. Now, we have:

$$\begin{aligned}
D_{n+2} = \sum_{k=0}^{m+1} \binom{(n+2)-k}{k} &= 1 + \sum_{k=1}^{m} \binom{(n+2)-k}{k} + \underbrace{\binom{(n+2)-(m+1)}{m+1}}_{=\binom{m+1}{m+1}=1} \\
&= 1 + \sum_{k=1}^{m} \left(\binom{(n+2)-1-k}{k} + \binom{(n+2)-1-k}{k-1} \right) + 1 && \text{(by Pascal's rule)} \\
&= 1 + \sum_{k=1}^{m} \binom{(n+1)-k}{k} + \sum_{k=1}^{m} \underbrace{\binom{(n+1)-k}{k-1}}_{=\binom{n-(k-1)}{k-1}} + 1 && \text{(linearity of the sum)} \\
&= \underbrace{\sum_{k=0}^{m} \binom{(n+1)-k}{k}}_{=D_{n+1}} + \sum_{k'=0}^{m-1} \binom{n-k'}{k'} + \underbrace{1}_{=\binom{m}{m}=\binom{n-m}{m}} && \text{(reindexing the second sum)} \\
&= D_{n+1} + \underbrace{\sum_{k'=0}^{m} \binom{n-k'}{k'}}_{=D_n} && \text{(gathering the 1 is the last sum)} \\
&= F(n+1) + F(n) && \text{(by induction hypothesis)} \\
&= F(n+2) && \text{(by definition of the Fibonacci numbers)}
\end{aligned}$$

  hence $D_{n+2} = F(n+2)$.

- If n is odd, let us write $n = 2m+1$ for some $m \in \mathbb{N}$. We observe that $\lfloor (n+2)/2 \rfloor = m+1$, $\lfloor (n+1)/2 \rfloor = m+1$ and $\lfloor n/2 \rfloor = m$. Now, we have:

$$\begin{aligned}
D_{n+2} = \sum_{k=0}^{m+1} \binom{(n+2)-k}{k} &= 1 + \sum_{k=1}^{m+1} \binom{(n+2)-k}{k} \\
&= 1 + \sum_{k=1}^{m+1} \left(\binom{(n+2)-1-k}{k} + \binom{(n+2)-1-k}{k-1} \right) && \text{(by Pascal's rule)} \\
&= 1 + \sum_{k=1}^{m+1} \binom{(n+1)-k}{k} + \sum_{k=1}^{m+1} \underbrace{\binom{(n+1)-k}{k-1}}_{=\binom{n-(k-1)}{k-1}} && \text{(linearity of the sum)} \\
&= \underbrace{\sum_{k=0}^{m+1} \binom{(n+1)-k}{k}}_{=D_{n+1}} + \underbrace{\sum_{k'=0}^{m} \binom{n-k'}{k'}}_{=D_n} && \text{(reindexing the second sum)} \\
&= F(n+1) + F(n) && \text{(by induction hypothesis)} \\
&= F(n+2) && \text{(by definition of the Fibonacci numbers)}
\end{aligned}$$

  hence $D_{n+2} = F(n+2)$.

  In all cases, we have shown that $D_{n+2} = F(n+2)$.

▹ *Conclusion* : by the induction principle, for all $n \in \mathbb{N}$, $D_n = F(n)$. By recalling the definition of D_n and $F(n)$, we have the wanted result: the sum of the n^{th} anti-diagonal in the Pascal's Triangle is the n^{th} Fibonacci number.

□